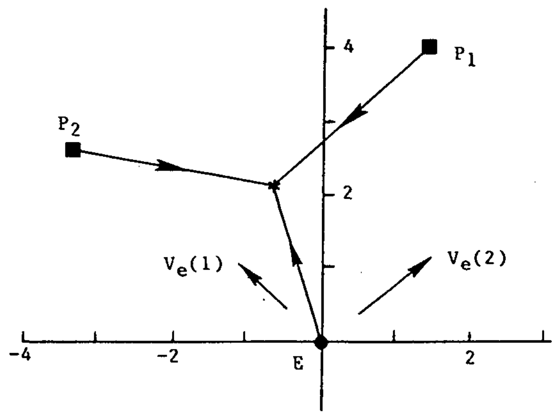


Fig. 3 Two-on-one optimization.

pursuer, additional complexities enter the problem. Figure 3 shows a two-on-one geometry. Here the circles of "reachable points" about each vehicle have an earliest common intersect time, and all three vehicles head toward that point. For higher dimensional interfering problems, the assignments and the tactics are less evident. For example, optimal evasion from one pursuer may mean that an evader moves toward a second pursuer. Such coupled trajectories remain an open question, even before real-space constraints are imposed.

## Discussion

Tactics are determined in the closed-loop sense, so suboptimal play by a single pursuer or evader could change the trajectories of all of the vehicles from that time forward. This was illustrated in Ref. 1 for a six-player example from football, for which the *evaders* were faster. Therefore, frequent position updates are required if each team is to profit from tactical errors of the other. Such errors may mean that assignments can change with time.

In addition to the potential complexities caused by trajectory interference or coupling, many practical features make the assignment problem more complex than simple examples suggest. For example, a detailed dynamic model near the time of intercept may be needed to answer the question, "Is this a hit or a near miss?"

Other operational refinements of potential importance could include differing noise levels in the position data of the vehicles, modification for spherical coordinates at long range, transient time intervals for maneuvers, range and angle limits of sensors, specification of discrete targets instead of a goal line, ranking the importance of such individual targets, varying the altitudes of individual vehicles, differing and variable speeds of individual vehicles, and real-space constraints.

## Conclusions

The assignment matrix formulation appears to be a practical approach to the solution of high-order versions of the team-intercept problem. If the heading and speed transients can be ignored, the total downrange gain is a performance index that can be optimized by the two teams. The guidance law is simple because the pursuers are assumed to be faster than the evaders, unless the geometry implies changes with time of the assignments.

## Acknowledgments

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## References

[1]Breakwell, J. V., and Merz, A. W., "Football as a Differential Game," AIAA Paper 90-3514, AIAA Guidance, Navigation, and Control Conference, Portland, OR, Aug. 20-22, 1990.

[2]Isaacs, R., *Differential Games—A Mathematical Theory of Warfare and Pursuit, Control and Optimization,* Wiley, New York, 1965, Chaps. 9, 10.

[3]Merz, A. W., "Implementing Air Combat Guidance Laws," *Journal of Dynamic Systems, Measurement and Control,* Vol. 111, No. 4, 1989, pp. 605-608.

[4]Austin, F., Carbone, G., Falco, M., Hinz, H., and Lewis, M., "Game Theory for Automated Maneuvering During Air-to-Air Combat," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 6, 1990, pp. 1143-1149.

[5]Guelman, M., Shinar, J., and Green, A., "Qualitative Study of a Planar Pursuit Evasion Game in the Atmosphere," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 6, 1990, pp. 1136-1142.

# Design of Low-Sensitivity Modalized Observers Using Left Eigenstructure Assignment

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## Introduction

A MULTIPLE objective optimization technique is applied to the design of modalized observers. The modalized observer was introduced by Andry et al.[1] and involves the choice of closed-loop observer left eigenvectors so that the effect of a known mismatch of initial conditions between the observer and plant is minimized. Such mismatches occur, for instance, in flight control problems when the aircraft (in a straight and level flight condition) is subject to a gust disturbance that introduces nonzero values to the variables angle of attack $\alpha$ and/or sideslip angle $\beta$.[1] To minimize the effect of such disturbances, Andry et al.[1] proposed a technique that involves the direct assignment[2] of the left eigenvectors of the observer to match a given prespecified set of vectors that results in a reduced state estimation error dynamic. However, as has been pointed out in Ref. 3, such a process is likely to produce an observer that possesses a set of eigenvalues that are highly sensitive to parameter variations or uncertainties that may occur in the system matrices.

A new approach was proposed[3] that involves the optimization of a multicriterion cost function that takes account of both eigenvalue sensitivity and of estimator error caused by an initial condition mismatch. This multiple objective approach involves the selection of a fixed set of closed-loop observer eigenvalues. The corresponding assignable eigenvector subspaces[2] are then calculated. This freedom for eigenvector assignment is used to reduce the value of the cost function using a quasi-Newton search with numerically evaluated gradients.

The present paper offers a slightly different approach that introduces several improvements over the work by Sobel and Banda.[3] An analysis, similar to that contained in Ref. 3, results in the definition of a new multiobjective cost function $J$ for which analytically derived gradients are available. This constitutes the first improvement since the convergence properties and accuracy of the solution using analytical gradients

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will be enhanced.[4] The newly defined cost function $J$ takes account of three desirable properties of an observer:

1) A certain known structure for a mismatch of initial conditions between observer and plant results in an attenuation of the state estimation error dynamics.

2) The observer exhibits low eigenvalue sensitivity to parameter variation or uncertainty.

3) The observer gain matrix is of low norm.

The final property was not considered in the cost function defined in Ref. 3 and is the second improvement of the new method. Property 3) is an important desirable property because, if an observer gain matrix has small entries (in modulus), then such an observer will be less sensitive to measurement error than one for which gains are large.

Each of the aforementioned properties are conflicting design specifications. For example, the lowest norm gain matrix that assigns a given set of eigenvalues will not, in general, be the design for which the error dynamic reduction is greatest and vice versa. Clearly, one must consider each of the desirable properties (1-3) at the design stage. A way to achieve this, as shown in this paper, is to optimize a scalar cost function that incorporates a term involving each of the specifications.

Weighting factors are introduced into the cost function $J$ so that more emphasis can be placed on the achievement of certain properties rather than others. This cannot (in its present form) be performed using the objective function in Ref. 3. The final improvement is given by allowing the closed-loop eigenvalues to be chosen from certain regions of the complex plane; more freedom is introduced into the optimization problem (solved using the Davidon-Fletcher-Powell method[4]), which facilitates the achievement of the design objectives.

The method is applied to the problem of gust alleviation for an aircraft.

## Problem Formulation

Consider the linear time-invariant state space representation of a system:

$$\dot{x} = Ax + Bu$$
$$y = Cx \tag{1}$$

where there are $n$ states, $m$ inputs, and $r$ outputs. The $(A,C)$ pair is assumed to be observable with rank $(C) = r$. Next, assume that a full state feedback design ($u = Kx$), by whatever method, has already been completed. This paper is concerned with the design of a full order observer, which possesses properties 1-3 given in the Introduction, represented by

$$\dot{z} = (A + HC)z - Hy + Bu \tag{2}$$

where $H$ is the observer gain matrix, which is to be specified, and $z$ is the estimate of the state vector $x$.

Define the error vector according to

$$e = z - x \tag{3}$$

The full closed-loop system (plant plus observer) may then be described by[5]

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BK & BK \\ 0 & A + HC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{4}$$

Consider the last $n$ rows of Eq. (4):

$$\dot{e} = (A + HC)e \tag{5}$$

The solution to Eq. (5) [provided $(A + HC)$ is nondefective[6]] may be expressed as[2]

$$e(t) = (L^{-1})^T e^{\Lambda t} L^T e(0) \tag{6}$$

where $L = [l_1, l_2, \ldots, l_n]$ is the left eigenvector matrix of $(A + HC)$, $e(0)$ is the initial condition of the error vector, $\Lambda = \mathrm{diag}(\lambda_i)$ and the $\lambda_i$ $(i = 1, 2, \ldots, n)$ are the eigenvalues of $(A + HC)$.

Taking the Frobenius norm of each side of Eq. (6) and squaring gives

$$\|e(t)\|_F^2 \le \|(L^{-1})^T\|_F^2 \, \|e^{\Lambda t}\|_F^2 \, \|L^T e(0)\|_F^2$$
$$\le n \, \|(L^{-1})^T\|_F^2 \, \|L^T e(0)\|_F^2 \; (= nJ_1) \tag{7}$$

If the right-hand side of Eq. (7) is minimized, then a bound on the error dynamic is reduced. For many flight control problems, information regarding the direction of the initial condition of the error vector is available.[1,3] The minimization of Eq. (7) alone will not produce, in general, an observer that has low eigenvalue sensitivity,[3] neither will the norm of the gain matrix necessarily be small. In order that the observer does satisfy these extra properties, it is necessary to define some measures in addition to that given in Eq. (7).

A measure of eigenvalue sensitivity to parameter variations or uncertainties in the system matrices is given, for the $i$th eigenvalue $\lambda_i$, by[6]

$$c_i = \frac{\|l_i\|_2 \, \|r_i\|_2}{|l_i^T r_i|} \ge 1 \tag{8}$$

where $l_i$ and $r_i$ are the corresponding $i$th left and right eigenvectors, respectively. Wilkinson[6] shows that these $c_i$ are bounded above:

$$\max_i c_i \le \|L^T\|_2 \, \|(L^{-1})^T\|_2 \; (= J_2) \tag{9}$$

Hence, if the right-hand side of Eq. (9) is small, then individual eigenvalue sensitivity is necessarily small.

Finally, the sum of the squared gain matrix elements is given by

$$J_3 = \|H\|_F^2 \tag{10}$$

The cost functions $J_1$, $J_2$, and $J_3$ give a direct measure, irrespective of the type of norm used, of properties 1-3 given in the Introduction. Therefore, if an overall cost function is given by

$$J = \alpha J_1 + \beta J_2 + \gamma J_3 \tag{11}$$

(where $\alpha$, $\beta$, and $\gamma$ are positive weighting factors), then a minimization of $J$ will produce an observer gain matrix $H$ that will satisfy properties 1-3. The extent to which the single-objective properties 1, 2, or 3 are satisfied will be determined by the choice of the weighting factors $\alpha$, $\beta$, and $\gamma$. Note that a vector cost function could be derived[4] and used as a replacement for Eq. (11) in the optimization process.

## First Derivative Calculation of the Objective Function

The partial derivatives of $J$ will be derived in this section. Only details of derivative calculations for real eigenvalues will be given—the extension to complex ones being straightforward.

Suppose that $\rho$ is a parameter of interest, then

$$\frac{\partial J}{\partial \rho} = \alpha \frac{\partial J_1}{\partial \rho} + \beta \frac{\partial J_2}{\partial \rho} + \gamma \frac{\partial J_3}{\partial \rho} \tag{12}$$

Equation (12) reveals that, trivially, the derivatives of the overall cost function $J$ are found by the addition of those for its constituent parts $J_1$, $J_2$, and $J_3$.

The derivatives of $J_1$ are first evaluated:

$$\frac{\partial J_1}{\partial \rho} = \frac{\partial}{\partial \rho} \left( \|(L^{-1})^T\|_F^2 \, \|L^T e(0)\|_F^2 \right) \tag{13}$$

$$= \frac{\partial}{\partial \rho}(\|(L^{-1})^T\|_F^2)\ \|L^T e(0)\|_F^2 + \|(L^{-1})^T\|_F^2 \frac{\partial}{\partial \rho}(\|L^T e(0)\|_F^2) \tag{14}$$

and hence

$$\frac{\partial J_1}{\partial \rho} = 2\left\{\text{trace}[L^{-1}(L^{-1})^T]\left[e(0)^T \frac{\partial L}{\partial \rho} L^T e(0)\right] + \text{trace}\left[\frac{\partial L^{-1}}{\partial \rho}(L^{-1})^T\right][e(0)^T L L^T e(0)]\right\} \tag{15}$$

The partial derivatives of $J_2$ are found via

$$\frac{\partial J_2}{\partial \rho} = \frac{\partial}{\partial \rho}[\|(L^{-1})^T\|_2\ \|L^T\|_2] = \frac{\partial}{\partial \rho}[\sigma_1(L^T)/\sigma_n(L^T)] \tag{16}$$

where $\sigma_i(L^T)$ is defined to be the $i$th singular value of $L^T$. The singular value decomposition[7] of a matrix $E \epsilon \mathbf{C}^{n \times n}$ is given by

$$E v_i = u_i \sigma_i$$

$$E^* u_i = v_i \sigma_i$$

$$v_i^* v_j = u_i^* u_j = \delta_{i,j} \text{(the Kronecker delta function)} \tag{17}$$

where $\sigma_i$ is the $i$th singular value of $E$, and $u_i$ and $v_i$ are the $i$th left and right singular vectors of $E$, respectively. Differentiating Eq. (17) partially with respect to $\rho$ and eliminating terms involving $\partial u_i/\partial \rho$ and $\partial v_i/\partial \rho$, we obtain

$$\frac{\partial \sigma_i(E)}{\partial \rho} = \text{real}\left(u_i^* \frac{\partial E}{\partial \rho} v_i\right) \tag{18}$$

The derivatives of $J_2$ are now easily computed according to

$$\frac{\partial J_2}{\partial \rho} = \left[\sigma_n(L^T)\frac{\partial \sigma_1(L^T)}{\partial \rho} - \sigma_1(L^T)\frac{\partial \sigma_n(L^T)}{\partial \rho}\right] \Big/ \sigma_n(L^T)^2 \tag{19}$$

The partial derivatives of $J_3$ have been found elsewhere[8] and hence are omitted from the analysis. To complete the derivative calculations, it is necessary to compute the terms $\partial L/\partial \rho$ and $\partial L^{-1}/\partial \rho$. It follows from the identity $LL^{-1} = I$, that

$$\frac{\partial L^{-1}}{\partial \rho} = -L^{-1}\frac{\partial L}{\partial \rho}L^{-1} \tag{20}$$

The matrix $L(=[l_1, l_2, \ldots, l_n])$ consists of the left eigenvectors $l_i$ for columns so that all that is required to be evaluated are the partial derivatives of $l_i$ with respect to the parameter $\rho$. The $i$th left eigenvector may actually be chosen by selecting the $r$th order design vector $w_i = (w_{i,1}, w_{i,2}, \ldots, w_{i,r})^T$ in the expression[2]

$$l_i = (\lambda_i I - A^T)^{-1} C^T w_i, \qquad (i = 1, 2, \ldots, n) \tag{21}$$

The eigenvector $l_i$ depends on $w_i$ and the particular choice of eigenvalue $\lambda_i$. Hence, if the eigenvalues may be chosen from regions of the complex plane, the parameter $\rho$ may be equal to $w_{i,j}$ or $\lambda_i$, and in either case

$$\frac{\partial L}{\partial \rho} = \left[\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \frac{\partial l_i}{\partial \rho}, \mathbf{0}, \ldots, \mathbf{0}\right] \tag{22}$$

The derivative with respect to $\lambda_i$ of $l_i$ is given by

$$\frac{\partial l_i}{\partial \lambda_i} = -(\lambda_i I - A^T)^{-2} C^T w_i \tag{23}$$

and for $w_{i,j}$

$$\frac{\partial l_i}{\partial w_{i,j}} = (\lambda_i I - A^T)^{-1} C^T e_j \tag{24}$$

where $e_j$ is an $r$th order zero vector with a unity element in the $j$th position.

If eigenvalues are allowed to be chosen from regions of the complex plane, a constrained optimization problem results. However, a transformation can be performed that produces an equivalent unconstrained problem provided the eigenvalues $\lambda_i = \lambda_{i,\text{re}} + \lambda_{i,\text{im}}$ are chosen from rectangular "regions of assignability" (see Ref. 8 for details).

## Example

Consider the linear equation of motion of the lateral dynamics of the L-1011 aircraft corresponding to a cruise flight condition as given in Ref. 1. The states, the first three of which are also the outputs, are roll angle, roll rate, yaw rate and sideslip angle. The controls are rudder deflection and aileron deflection. The control law ($u = Kx$) based on full state information is given in Ref. 3.

Assume in the following that $t$ is large. Hence, for a stable observer $z(t) \simeq x(t)$, and if the controlled system is stable, this implies that $e(t) \simeq z(t) \simeq x(t) \simeq 0$. Suppose that such a condition is attained and (without loss of generality) that $t = 0$ at the time of a gust disturbance. The initial condition of the state and the observer will be given by[1,3]

$$x(0) = -e(0) \propto (0, 0, 0, 1)^T \tag{25}$$

Using this a priori information on the structure of the disturbance, a design will be performed (using the new method) that is to be compared with the design obtained by Sobel and Banda.[3] It is required that the impact of a mismatch of the form given in Eq. (25) is minimized and, supplementary to this, that the observer system possesses eigenvalues that are insensitive to parameter variation or uncertainty and that the design is achieved with a small (in norm) gain matrix.

The fixed set of eigenvalues ($\lambda_1 = -3$, $\lambda_2 = -3.5$, $\lambda_3 = -4$, $\lambda_4 = -4.5$) is assigned. The weighting factors are chosen to place emphasis on the reduction of $J_1$ in the objective function so that we choose $\alpha = 5$, $\beta = 1$, and $\gamma = 1$. The resultant gain matrix is given by

$$H_1 = \begin{bmatrix} -3.5026 & -0.9855 & 0.0331 \\ 0.0027 & -4.5606 & 1.5289 \\ 0.0384 & 1.1072 & -1.8568 \\ -0.0230 & 1.5998 & -0.2809 \end{bmatrix} \tag{26}$$

A comparison of the values of the single-objective cost functions $J_1$, $J_2$, and $J_3$ for $H_1$ and the gain matrix ($H_S$, say) provided by Sobel and Banda[3] are given in Table 1. Also given are the values of the individual eigenvalue condition numbers ($c_i$) defined in Eq. (8). Define $c = (c_1, c_2, c_3, c_4)^T$. Both the error vector attenuation factor (characterized by $J_1$) and the overall eigenvalue conditioning (given by $\|c\|_2$), for the design produced in this paper, are improved relative to the design produced by Sobel and Banda. Also, the Frobenius norm of the gain matrix $H_1$ is reduced relative to $H_S$. The improvement in both $J_1$ and $J_2$ indicates that the gain matrix produced in Ref. 3 is nonoptimal. This is probably because, in that work, only numerically evaluated gradients were used or, alternatively, that the optimization was not rigorously pursued. Also, the cost function used in the optimization described in this paper is different from that of Sobel and Banda.[3] Interestingly, the two-norm of $H_1$ ( = 5.48) is increased relative to that of the old design $H_S$ ( = 5.42). This may occur because it is the Frobenius norm of the gain matrix that is optimized. Actually, one might reasonably expect a reduction in the two-norm of the gain matrix for two reasons. First, because Kautsky et al.[9] have shown that the two-norm of the gain is bounded above by a quantity involving $J_2$. Hence, reducing the eigenvalue sensitivities will reduce a bound on the two-norm of the gain. Second, the two-norm of the gain matrix is also bounded above by its Frobenius norm.[7] However, this example is an

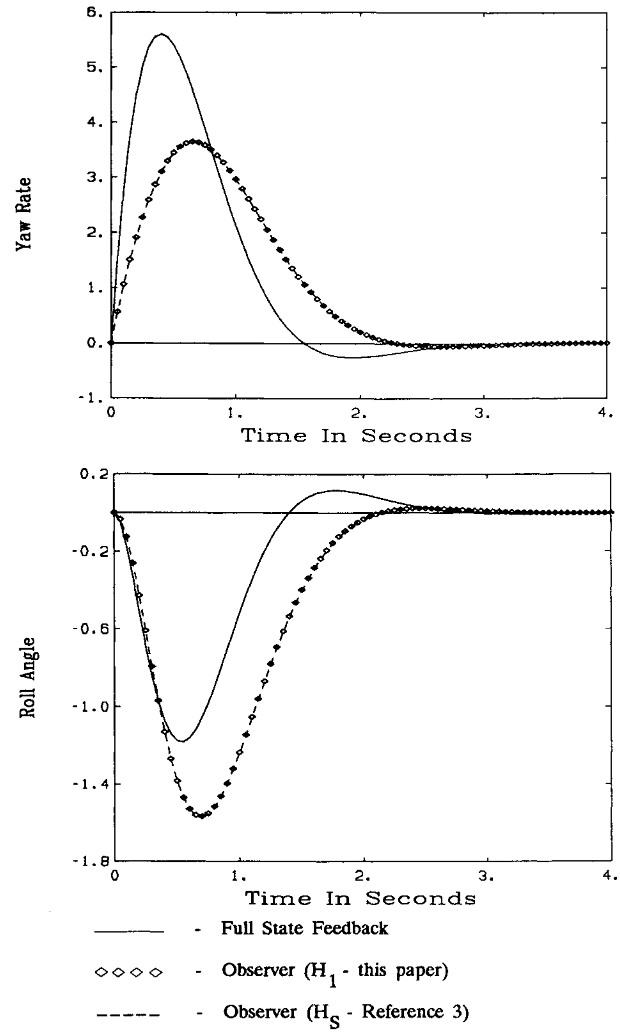


**Fig. 1 Actual aircraft states for each control law.**

**Table 1 Comparison of the gain matrices $H_1$ and $H_S$**

| Property | $H_1$ (this paper) | $H_S$ (Ref. 3) |
|---|---|---|
| $J_1^{1/2}$ | 12.30 | 13.44 |
| $J_2$ | 10.95 | 11.23 |
| $J_3^{1/2}$ | 6.61 | 6.74 |
| $c_1$ | 5.49 | 5.62 |
| $c_2$ | 1.00 | 3.07 |
| $c_3$ | 1.01 | 3.47 |
| $c_4$ | 5.49 | 5.39 |
| $\lVert c \rVert_2$ | 7.89 | 9.06 |

illustration of the fact that reducing a bound on the norm of the gain matrix does not imply that this norm will be reduced.

For a gust-induced initial sideslip, let the initial condition of the state vector be $x(0) = (0,\ 0,\ 0,\ 5)^T[= -e(0)]$. Time responses of some of the actual states of the aircraft are given in Fig. 1 for the full state control law and each of the systems formed by using the observer gain matrices $H_1$ and $H_S$. Note that the closed-loop systems including the observers exhibit an almost identical time response with gain matrices of similar norm. However, the design using the method introduced in this paper ($H_1$) possesses reduced overall eigenvalue sensitivity compared to the old design ($H_S$).

A design with reduced eigenvalue sensitivity or with a lower norm gain matrix, etc., may be obtained by allowing the closed-loop eigenvalues to be selected in conjugate pairs from regions of the complex plane. Alternatively, one could place more emphasis on certain design objectives by altering the weighting factors $\alpha$, $\beta$, and $\gamma$.

## Conclusion

A multiple objective optimization technique using analytically derived gradients has been described. This technique allows for the design of an observer that has low eigenvalue sensitivity to parameter variation or uncertainty in the system matrices and is achieved using a small (in norm) gain matrix that is of reduced sensitivity to measurement error. Supplementary to this, the impact of a known structure (or direction) for an initial condition mismatch is reduced by assigning the left eigenstructure of the observer.

The new method provides improved results over former methods because of the use of analytically derived derivatives of the multiple objective cost function. The earlier methods used numerically derived gradients that may sometimes provide a non-optimal solution. In addition, provision is made so that the design freedom may be increased by allowing closed-loop eigenvalues to be chosen from regions of the complex plane.

One is able to place more (or less) emphasis on the single objectives contained in the overall cost function by the adjustment of certain weighting factors that have been introduced.

## References

[1]Andry, A. N., Chung, J. C., and Shapiro, E. Y., "Modalized Observers," *IEEE Transactions on Automatic Control*, Vol. 29, No. 7, 1984, pp. 669–671.

[2]Andry, A. N., Shapiro, E. Y., and Chung, J. C., "Eigenstructure Assignment for Linear Systems," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 19, No 5, 1983, pp. 711–729.

[3]Sobel, K. M., and Banda, S. S., "Design of a Modalized Observer with Eigenvalue Sensitivity Reduction," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 5, 1989, pp. 762–763.

[4]Gill, P. E., Murrary, W., and Wright, M. H., *Practical Optimization*, Academic Press, New York, 1981.

[5]Franklin, G. F., Powell, J. D., and Emami-Naeini, A., *Feedback Control of Dynamic Systems*, Addison-Wesley, Reading, MA, 1986.

[6]Wilkinson, J. H., *The Algebraic Eigenvalue Problem*, Oxford University Press, U.K., 1965.

[7]Golub, G. H., and Van Loan, C. F., *Matrix Computations*, North Oxford Academic, 1983.

[8]Roppenecker, G., "Minimum Norm Output Feedback Design Under Specified Eigenvalue Areas," *Systems & Control Letters*, Vol. 3, No. 2, 1983, pp. 101–103.

[9]Kautsky, J., Nichols, N. K., and Van Dooren, P., "Robust Pole Assignment in Linear State Feedback," *International Journal of Control*, Vol. 41, No. 5, 1985, pp. 1129–1155.

# Pulse Response Method for Vibration Reduction in Periodic Dynamic Systems

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## Introduction

SEVERAL periodic dynamic systems possess unwanted vibration, including the helicopter, which has received much attention recently. Periodic systems cannot be described in terms of classical transfer functions or time-invariant state transition matrices. Thus, conventional control theory, such

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